Received: *June 25, 1999*, Accepted: *August 27, 1999*

On the logarithmic behaviour in $\mathcal{N} = 4$ SYM theory

Massimo Bianchi, Stefano Kovacs, Giancarlo Rossi and Yassen S. Stanev*

Dipartimento di Fisica, Università di Roma "Tor Vergata"
I.N.F.N. – Sezione di Roma "Tor Vergata"
Via della Ricerca Scientifica 1, 00173 Roma, Italy
E-mail: massimo.bianchi@roma2.infn.it, Stefano.Kovacs@roma2.infn.it, rossig@roma2.infn.it

Abstract: We show that the logarithmic behaviour seen in perturbative and non perturbative contributions to Green functions of gauge-invariant composite operators in $\mathcal{N} = 4$ SYM with SU(N) gauge group can be consistently interpreted in terms of anomalous dimensions of unprotected operators in long multiplets of the superconformal group SU$(2,2|4)$. In order to illustrate the point we analyse the short-distance behaviour of a particularly simple four-point Green function of the lowest scalar components of the $\mathcal{N} = 4$ supercurrent multiplet. Assuming the validity of the Operator Product Expansion, we are able to reproduce the known value of the one-loop anomalous dimension of the single-trace operators in the Konishi supermultiplet. We also show that it does not receive any non-perturbative contribution from the one-instanton sector. We briefly comment on double- and multi-trace operators and on the bearing of our results on the AdS/SCFT correspondence.

Keywords: Extended Supersymmetry, Duality in Gauge Field Theories.

*On leave of absence from Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, BG-1784, Sofia, Bulgaria.

Contents

1. Introduction and summary of the results

As is well known, conformal invariance puts tight constraints on the correlation functions of primary operators [1]. In particular, two- and three-point Green functions are fixed up to multiplicative constants. Four-point functions in four-dimensional Conformal Field Theories (CFT's), however, are determined by conformal invariance only up to *a priori* unknown functions of two independent conformal invariant cross-ratios. As a consequence the comparison of four-point amplitudes computed in type IIB supergravity/superstring theory in $AdS_5 \times S^5$ with four-point correlation functions in $\mathcal{N} = 4$ super Yang–Mills theory (SYM) represents a truly 'dynamical' and highly non-trivial check of the correspondence proposed by Maldacena [2].

A dynamical check of the Maldacena conjecture was indirectly carried out for some higher-derivative terms in the AdS effective action in [3], where even in the case of an SU(2) gauge group the non-perturbative (one-instanton) correction to a sixteen-fermion Green function was shown to precisely match the result of a similar non-perturbative (one-D-instanton) correction in the AdS supergravity theory. A supersymmetry related four-point function of scalar composite operators belonging to the $\mathcal{N} = 4$ supercurrent multiplet was also explicitly computed. This matching was recently proven to hold for any number of colours, N, and for any instanton number in the large N limit [4].

A common feature of all four-point Green function computations performed so far, both in the AdS description [5, 6, 7, 8] and in $\mathcal{N} = 4$ SYM theory [9, 10], is the appearance of short-distance logarithmic singularities. In this paper we demonstrate that in the conformal phase of SYM theory this behaviour can be explained as a result of the presence of operators with non-vanishing anomalous dimensions.

Indeed only short supermultiplets of the global SU$(2,2|4)$ superconformal symmetry are protected from receiving quantum corrections to their scale dimensions, Δ [11]. The lowest scalar components of a short multiplet belong to a representation of highest weight $[0,\ell,0]$ of the SU(4) R-symmetry and have scale dimension $\Delta = \ell$, for some integer $\ell \geq 2$ [12]. A familiar example is the $\mathcal{N} = 4$ supercurrent multiplet with $\ell = 2$, whose AdS counterpart is the gauged supergravity multiplet [13].

Long multiplets on the contrary can exist for any value of the scale dimension of their lowest component [11]. The presence of an infinite tower of long supermultiplets is needed in order to account for causality and locality in the AdS/SCFT correspondence [14]. A well-known example of a long supermultiplet is the Konishi supermultiplet [15] that involves single (colour) trace operators. In the large N limit the Konishi-type supermultiplets, that correspond to genuine string excitations [11], are expected to receive (large) corrections to their anomalous dimensions [16, 17].

There exist also multi-trace operators that may belong both to short and to long multiplets [11]. The former are protected while, as we will show in this paper, the corrections to the anomalous dimensions of the latter are zero or vanishingly small in the large N limit.

In order to illustrate our point, in this paper we concentrate on the simplest example of unprotected operator, namely the lowest scalar component of the Konishi multiplet, which has naive scale dimension $\Delta^{(0)} = 2$. We compute both the one-loop and the one-instanton contributions to a particular four-point function in which this operator is exchanged in only one of the intermediate channels. Assuming the validity of the Operator Product Expansion (OPE), we reproduce the known value of the one-loop anomalous dimension of the Konishi multiplet as a non trivial check of our interpretation of the appearence of short-distance logarithmic singularities. Moreover, the analysis of the non-perturbative contribution demonstrates that both the anomalous dimension of the Konishi multiplet and its trilinear OPE coefficients with operators in the supercurrent multiplet receive only perturbative corrections.

The outline of the paper is as follows. After a preliminary discussion on anomalous dimensions in a generic CFT in section 2, in section 3 we identify a simple four-point function that allows us to most directly expose and interpret the short-distance logarithmic behaviour in $\mathcal{N} = 4$ SYM theory. One-loop and one-instanton computations are described in sections 4 and 5, respectively. Section 6 contains our conclusions and comments on the logarithmic behaviour found in genuine AdS computations.

2. Anomalous dimensions in Conformal Field Theory

Anomalous dimensions usually (e.g. in QCD) appear as a consequence of the need of regularising and renormalising the theory. At first sight their appearance might seem surprising in a theory, such as SU(N) $\mathcal{N} = 4$ SYM, that is 'known' to be finite [18].

Computations performed so far have indeed shown that four-point functions of composite operators belonging to the supercurrent multiplet are finite at non-coincident points. On the other hand, even two-point functions of protected operators, that are expected not to be corrected in perturbation theory nor after inclusion of instanton effects, are not finite in a distributional sense. In order to make them finite one has to subtract short-distance non-integrable singularities [19]. This, however, is necessary even in a free theory and has nothing to do with the non-trivial dynamics of SU(N) $\mathcal{N}=4$ SYM.

To illustrate in a simple fashion the emergence of logarithmic terms and their relation to anomalous dimensions, let us consider the two-point function of a primary operator of scale dimension Δ

$$\langle \mathcal{O}^{\dagger}_{\Delta}(x)\mathcal{O}_{\Delta}(y)\rangle = \frac{A_{\Delta}}{(x-y)^{2\Delta}}\,, \tag{2.1}$$

where A_{Δ} is an overall normalisation constant possibly depending on the subtraction scale μ. Now suppose that $\Delta = \Delta^{(0)} + \gamma$, i.e. the operator under consideration has an anomalous dimension. In perturbation theory $\gamma = \gamma(g)$ is expected to be small and to admit an expansion in the coupling constant g. The perturbative expansion of (2.1) in powers of γ yields

$$\langle \mathcal{O}^{\dagger}_{\Delta}(x)\mathcal{O}_{\Delta}(y)\rangle = \frac{a_{\Delta}}{(x-y)^{2\Delta^{(0)}}}\left(1-\gamma\log\left[\mu^2(x-y)^2\right]+\frac{1}{2}\gamma^2\left(\log[\mu^2(x-y)^2]\right)^2+\cdots\right), \tag{2.2}$$

where after renormalisation we have set $A_{\Delta} = a_{\Delta}\mu^{-2\gamma}$. Thus, although the exact expression (2.1) is conformally invariant, at each order in γ (or in g) (2.2) contains logarithms that seem to even violate scale invariance. Let us stress that the appearance of logarithmic terms is an artifact of the perturbative expansion, rather than an intrinsic property of the theory.

Similar considerations apply to generic n-point Green functions as well. Assuming the validity of the OPE, a four-point function can be expanded in the s-channel in the form

$$\begin{aligned}\langle \mathcal{Q}_A(x)\mathcal{Q}_B(y)\mathcal{Q}_C(z)\mathcal{Q}_D(w)\rangle &= \\ &= \sum_K \frac{{C_{AB}}^K(x-y,\partial_y)}{(x-y)^{\Delta_A+\Delta_B-\Delta_K}}\frac{{C_{CD}}^K(z-w,\partial_w)}{(z-w)^{\Delta_C+\Delta_D-\Delta_K}}\langle \mathcal{O}_K(y)\mathcal{O}_K(w)\rangle\,,\end{aligned} \tag{2.3}$$

where K runs over a (possibly infinite) complete set of primary operators. Descendants are implicitly taken into account by the presence of derivatives in the Wilson coefficients, C's. An expansion like (2.3) is valid in the other two channels as well. To simplify formulae we assume that $\mathcal{Q}_A$, $\mathcal{Q}_B$, $\mathcal{Q}_C$, $\mathcal{Q}_D$ are protected operators, i.e. they have no anomalous dimensions. In general the operators $\mathcal{O}_K$ may have anomalous dimensions, γ_K, so that $\Delta_K = \Delta_K^{(0)} + \gamma_K$, where $\Delta_K^{(0)}$ is the tree-level scale dimension. Similarly $C_{IJ}{}^K = C_{IJ}^{(0)\,K} + \eta_{IJ}{}^K$, with $\eta_{IJ}{}^K$ the perturbative correction to the OPE coefficients. Indeed, although three-point functions of single-trace protected operators are known not to be renormalised beyond tree level [20], *a priori* nothing can be said concerning corrections to three-point functions involving also unprotected operators.

Neglecting descendants, one gets for the first-order terms of (2.3) in the small parameters γ and η

$$\langle \mathcal{Q}_A(x)\mathcal{Q}_B(y)\mathcal{Q}_C(z)\mathcal{Q}_D(w)\rangle_{(1)} = \sum_K \frac{\langle \mathcal{O}_K(y)\mathcal{O}_K(w)\rangle_{(0)}}{(x-y)^{\Delta_A+\Delta_B-\Delta_K^{(0)}}(z-w)^{\Delta_C+\Delta_D-\Delta_K^{(0)}}} \times$$
$$\times \left[\eta_{AB}{}^K C_{CD}^{(0)\,K} + C_{AB}^{(0)\,K}\eta_{CD}{}^K + \frac{\gamma_K}{2} C_{AB}^{(0)\,K} C_{CD}^{(0)\,K} \log\frac{(x-y)^2(z-w)^2}{(y-w)^4}\right]. \quad (2.4)$$

From this perturbative formula one can extract the corrections to both the OPE coefficients and the anomalous dimensions of the operators $\mathcal{O}_K$. The former come from the terms in (2.4) that display the same singularities as dictated by naive dimensional analysis, the latter from the coefficients of the logarithmic terms. Similar considerations have been applied to $\mathrm{O}(N)$ models in $2 < D < 4$ in order to compute $1/N$ corrections to anomalous dimensions [21].

We end this section with a general remark on anomalous dimensions. Positivity of two point-functions in the distributional sense implies lower bounds on the anomalous dimension of the operators. In particular, scale dimensions of rank r symmetric tensors have to satisfy $\Delta \geq 2 + r$ [22]. For a scalar field the bound is $\Delta \geq 1$, which is obviously saturated by a free field. If, however, the scalar field belongs to a supermultiplet that also contains some tensor field, there are additional constraints coming from the relation imposed by supersymmetry between the scale dimensions of scalar and tensor fields [23].

3. Tree-level considerations

In this section we identify a simple four-point function of protected operators that involves the exchange of unprotected operators and allows a direct determination of the lowest order corrections to their anomalous dimensions and OPE coefficients. The choice obviously falls on the operators in the $\mathcal{N} = 4$ current multiplet. They play a central rôle in the correspondence with type IIB superstring theory on $AdS_5 \times S^5$, since they couple to the fields of the gauged supergravity multiplet. Differently from AdS-inspired computations, where the scalar singlets of the SU(4) R-symmetry in the dilaton-axion sector are amenable to explicit computations [8], we prefer to work, as in [3, 4] and in [20], with the lowest scalar components in the current supermultiplet, $\mathcal{Q}_{\mathbf{20}}^{ij}$, that belong to the representation **20** of SU(4). In terms of the six real scalars

φ^i belonging to the representation **6** of SU(4), they are defined as

$$\mathcal{Q}^{ij}_{\mathbf{20}} = \mathrm{tr}\left(\varphi^i\varphi^j - \frac{\delta^{ij}}{6}\varphi_k\varphi^k\right) . \tag{3.1}$$

The trace over the SU(N) colour indices, $a, b = 1, \dots , N^2 - 1$, is defined as usual by

$$\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab} , \tag{3.2}$$

where T^a are the generators in the fundamental representation of the SU(N) gauge group. In order to fix our normalisations we write down explicitly the two-point function of the scalar fields, which reads

$$\langle\varphi^{ia}(x_1)\varphi^{jb}(x_2)\rangle = \frac{1}{(2\pi)^2}\frac{\delta^{ij}\delta^{ab}}{x_{12}^2} , \tag{3.3}$$

where $x_{pq} = x_p - x_q$.

In order to analyse the short-distance behaviour of 4-point functions, it is convenient to start from the OPE of two $\mathcal{Q}_{\mathbf{20}}$. At tree level it takes the form

$$\begin{aligned}\mathcal{Q}_{\mathbf{20}}(x_1)\mathcal{Q}_{\mathbf{20}}(x_2) = {} & \frac{N^2-1}{(2\pi)^4x_{12}^4} + \frac{1}{(2\pi)^2x_{12}^2}\left[\mathcal{Q}_{\mathbf{20}}\left(\frac{x_1+x_2}{2}\right) + \mathcal{K}_{\mathbf{1}}\left(\frac{x_1+x_2}{2}\right)\right] + \\ & + \frac{x_{12\mu}}{(2\pi)^2x_{12}^2}\left[\mathcal{J}^\mu_{\mathbf{15}}\left(\frac{x_1+x_2}{2}\right) + \mathcal{K}^\mu_{\mathbf{15}}\left(\frac{x_1+x_2}{2}\right)\right] + \cdots , \end{aligned} \tag{3.4}$$

where track has been kept of primary fields only and all the SU(4) indices are omitted. The dots represent operators of (tree-level) scale dimension $\Delta^{^{(0)}} \geq 4$. Bold subscripts denote the SU(4) representations to which the various operators belong. Besides the identity operator, there appear protected as well as *a priori* unprotected operators. Among the protected single-trace operators one finds $\mathcal{Q}_{\mathbf{20}}$ itself with $\Delta = 2$, and the R-symmetry current $\mathcal{J}^\mu_{\mathbf{15}}$ with $\Delta = 3$. Among the unprotected single-trace operators one finds operators belonging to the long Konishi multiplet, which we denote by $\mathcal{K}$. $\mathcal{K}$ starts with the SU(4) singlet scalar of naive dimension $\Delta^{^{(0)}} = 2$

$$\mathcal{K}_{\mathbf{1}} = \frac{1}{3} : \mathrm{tr}\left(\varphi^i\varphi^i\right) : \tag{3.5}$$

and includes also the classically conserved current, $\mathcal{K}^\mu_{\mathbf{15}}$ with $\Delta^{^{(0)}} = 3$. As discussed at the end of section 2, positivity constraints on anomalous dimensions imply the bound $\Delta \geq 2$ for the dimension of $\mathcal{K}_{\mathbf{1}}$.

In order to illustrate the origin and the rôle of short distance logarithmic singularities in the simplest way, it is convenient to consider the four-point Green function

$$G^{(H)}(x_1, x_2, x_3, x_4) = \langle\mathcal{C}^{11}(x_1)\mathcal{C}^\dagger_{11}(x_2)\mathcal{C}^{22}(x_3)\mathcal{C}^\dagger_{22}(x_4)\rangle , \tag{3.6}$$

where we have introduced the gauge-invariant composite operators

$$\mathcal{C}^{IJ} = \mathrm{tr}(\phi^I \phi^J)\,, \qquad \mathcal{C}^\dagger_{IJ} = \mathrm{tr}(\phi^\dagger_I \phi^\dagger_J)\,. \tag{3.7}$$

In the previous equation we have defined the complex elementary fields

$$\phi^I = \frac{1}{\sqrt{2}}\left(\varphi^I + i\varphi^{I+3}\right), \qquad \phi^\dagger_I = \frac{1}{\sqrt{2}}\left(\varphi^I - i\varphi^{I+3}\right), \qquad I = 1,2,3 \tag{3.8}$$

that belong to the representations $\mathbf{3}_{+1}$ and $\mathbf{3}_{-1}$ in the decomposition of the representation $\mathbf{6}$ of SU(4) with respect to SU(3) $\times$ U(1).

At tree-level $G^{(H)}$ contains only disconnected diagrams and has the expression

$$G^{(H)}_{(0)}(x_1,x_2,x_3,x_4) = \frac{(N^2-1)^2}{4(2\pi)^8 x_{12}^4 x_{34}^4}\,. \tag{3.9}$$

From the point of view of the OPE (3.4) the x-dependence of eq. (3.9) may appear very surprising, because one would naively expect that intermediate operators of dimension 2, 3 etc. should contribute giving rise to additional subdominant singularities in x_{pq}. The point is that the Green function (3.6) can be expressed as the product of the two OPE's

$$\mathcal{C}^{11}(x)\mathcal{C}^\dagger_{11}(y) = \frac{N^2-1}{2(2\pi)^4(x-y)^4} + \frac{1}{(2\pi)^2(x-y)^2}\left(\mathcal{K}_{\mathbf{1}} + \mathcal{Q}^{(Y)}_{\mathbf{20}} + \mathcal{Q}^{(X)}_{\mathbf{20}}\right)\left(\frac{x+y}{2}\right) + \cdots \tag{3.10}$$

and

$$\mathcal{C}^{22}(x)\mathcal{C}^\dagger_{22}(y) = \frac{N^2-1}{2(2\pi)^4(x-y)^4} + \frac{1}{(2\pi)^2(x-y)^2}\left(\mathcal{K}_{\mathbf{1}} + \mathcal{Q}^{(Y)}_{\mathbf{20}} - \mathcal{Q}^{(X)}_{\mathbf{20}}\right)\left(\frac{x+y}{2}\right) + \cdots, \tag{3.11}$$

where

$$\mathcal{Q}^{(Y)}_{\mathbf{20}} = \frac{1}{3}\mathrm{tr}\left(\phi^\dagger_1\phi^1 + \phi^\dagger_2\phi^2 - 2\phi^\dagger_3\phi^3\right) \tag{3.12}$$

$$\mathcal{Q}^{(X)}_{\mathbf{20}} = \mathrm{tr}\left(\phi^\dagger_1\phi^1 - \phi^\dagger_2\phi^2\right) \tag{3.13}$$

$$\mathcal{K}_1 = \frac{1}{3} : \mathrm{tr}(\varphi^i\varphi^i) : = \frac{2}{3} : \mathrm{tr}(\phi^\dagger_I\phi^I) : \tag{3.14}$$

and cancellations among the contributions of different operators take place. For example, the pole $1/x_{12}^2$ in the limit $x_{12}\to 0$ (s-channel) is absent, since the contribution of the operator $\mathcal{Q}^{(X)}_{\mathbf{20}}$ exactly cancels the contributions of $\mathcal{Q}^{(Y)}_{\mathbf{20}}$ and $\mathcal{K}_{\mathbf{1}}$ to the four-point function (3.9). Note that this cancellation is not possible at higher orders, since $\mathcal{K}_{\mathbf{1}}$ has an anomalous dimension while $\mathcal{Q}_{\mathbf{20}}$ is protected.

Similar tree-level cancellations take place among the spin one currents. For instance, the potential pole x^μ_{12}/x^2_{12} has vanishing coefficient because the components

JHEP08(1999)020

of the currents, $\mathcal{K}^\mu_{\mathbf{15}}$ and $\mathcal{J}^\mu_{\mathbf{15}}$, that couple to $\mathcal{C}^{11}(x)\mathcal{C}^\dagger_{11}(y)$ are orthogonal to the ones that couple to $\mathcal{C}^{22}(x)\mathcal{C}^\dagger_{22}(y)$. Notice that this is true separately for $\mathcal{K}^\mu_{\mathbf{15}}$ and $\mathcal{J}^\mu_{\mathbf{15}}$, so that this cancellation will survive radiative corrections despite the fact that they induce an anomalous dimension for $\mathcal{K}^\mu_{\mathbf{15}}$ but not for $\mathcal{J}^\mu_{\mathbf{15}}$. Anticipating the results of the next section, let us note that at one loop the only field of tree-level dimension $\Delta^{(0)} \leq 10$ with non vanishing contribution to the s-channel of the function (3.6) is the lowest Konishi scalar $\mathcal{K}_{\mathbf{1}}$. This demonstrates that reading off the complete operator content from a given four-point function might be rather subtle.

Let us briefly comment on the OPE in the other two channels of $G^{(H)}_{(0)}$ in (3.9). The limit $x_{13} \to 0$ in (3.6) exposes the u-channel (the t-channel is similar) and singles out the double-trace operator

$$\mathcal{D}^{(11|22)} = \mathcal{C}^{11}\mathcal{C}^{22}\,. \tag{3.15}$$

This operator is automatically normal ordered and belongs to the reducible $\mathbf{105}+\mathbf{84}$ representation of SU(4). In fact one has

$$\mathcal{D}^{(11|22)}_{\mathbf{105}} = \frac{1}{3}\left(\mathcal{C}^{11}\mathcal{C}^{22} + 2\mathcal{C}^{12}\mathcal{C}^{12}\right), \tag{3.16}$$

$$\mathcal{D}^{(11|22)}_{\mathbf{84}} = \frac{1}{2}\left(\mathcal{C}^{11}\mathcal{C}^{22} - \mathcal{C}^{12}\mathcal{C}^{12}\right).. \tag{3.17}$$

As indicated, the operator $\mathcal{D}^{(11|22)}_{\mathbf{105}}$ belongs to the $\mathbf{105}$ representation of SU(4) whose highest weight is $[0,4,0]$. Since its naive dimension, $\Delta^{(0)}$, is exactly 4, $\mathcal{D}^{(11|22)}_{\mathbf{105}}$ is the lowest scalar component of a short and thus protected supermultiplet. Without spoiling the $SU(2,2|4)$ superconformal invariance, the supermultiplet that starts with $\mathcal{D}_{\mathbf{105}}$ can be made orthogonal to the short supermultiplet associated to the quartic Casimir that starts with the scalar composite operator

$$\mathcal{Q}_{\mathbf{105}} = \mathrm{tr}(\phi^4)\,. \tag{3.18}$$

In particular it is convenient to define the combination

$$\widehat{\mathcal{Q}}_{\mathbf{105}} \equiv \mathcal{Q}_{\mathbf{105}} - \frac{\langle \mathcal{D}^\dagger_{\mathbf{105}}\mathcal{Q}_{\mathbf{105}}\rangle}{\langle \mathcal{D}^\dagger_{\mathbf{105}}\mathcal{D}_{\mathbf{105}}\rangle}\mathcal{D}_{\mathbf{105}} = \mathrm{tr}(\phi^4) - \frac{2N^2-3}{N(N^2+1)}[\mathrm{tr}(\phi^2)]^2\,, \tag{3.19}$$

that vanishes for $N=2$ i.e. for an SU(2) gauge group. Notice that with this definition $\widehat{\mathcal{Q}}_{\mathbf{105}}$ has zero three-point function with two $\mathcal{Q}_{\mathbf{20}}$, for any N. As a consequence the operator $\widehat{\mathcal{Q}}_{\mathbf{105}}$ cannot contribute to the short distance limits that we study. The only operator in the representation $\mathbf{105}$ that is relevant for our analysis is thus $\mathcal{D}_{\mathbf{105}}$. In the next sections we will argue that $\mathcal{D}_{\mathbf{105}}$ does not receive quantum corrections to its naive scale dimension and OPE coefficients. The above observation about the decoupling of $\widehat{\mathcal{Q}}_{\mathbf{105}}$ is in line with its AdS interpretation as a Kaluza-Klein excitation ($\ell = 4$) of $\mathcal{Q}_{\mathbf{20}}$ and clarifies a point raised in [20, 24].

JHEP08(1999)020

The operator $\mathcal{D}_{\mathbf{84}}^{(11|22)}$ belongs to a long supermultiplet instead since the highest weight of the $\mathbf{84}$ has Dynkin labels $[2,0,2]$, while the naive dimension of $\mathcal{D}^{(11|22)}$ is $\Delta^{(0)} = 4$. The operator $\mathcal{D}_{\mathbf{84}}^{(11|22)}$ mixes with

$$\mathcal{K}_{\mathbf{84}}^{(11|22)} = g^2 \mathrm{tr}\left([\phi^1, \phi^2][\phi^1, \phi^2]\right) , \tag{3.20}$$

that lies in the $\mathbf{84}$ of SU(4) and belongs to the Konishi supermultiplet.[1] The mixing is maximal for SU(2) in which case $\mathcal{K}_{\mathbf{84}}$ and $\mathcal{D}_{\mathbf{84}}$ are proportional to one another. In general it is convenient to define an operator $\widehat{\mathcal{D}}_{\mathbf{84}}$ that has vanishing two-point function at tree-level with $\mathcal{K}_{\mathbf{84}}$

$$\widehat{\mathcal{D}}_{\mathbf{84}} = \mathcal{D}_{\mathbf{84}} - \frac{1}{g^2 N}\mathcal{K}_{\mathbf{84}} . \tag{3.21}$$

At higher order one should appropriately adjust the form of the linear combinations that diagonalise the two-point functions in order to construct operators with well-defined anomalous dimensions.

Considerations of $\mathcal{N}$=1 superconformal symmetry suggest that $\mathcal{D}_{\mathbf{84}}^{(11|22)}$ should be a chiral primary operator and as such it should have vanishing anomalous dimension.[2] The shortening conditions for general $\mathrm{SU}(2,2|\mathcal{N})$ supermultiplets are discussed in [25, 26]. Our results are consistent with $\gamma_{\widehat{\mathcal{D}}_{\mathbf{84}}} = 0$ both at the perturbative and at the non-perturbative level.

4. One-loop perturbative calculations

In perturbative calculations it is necessary to work with an off-shell formulation, so that an $\mathcal{N} = 4$ superfield approach, which admits only an on-shell description, cannot be employed. The use of $\mathcal{N} = 1$ superspace techniques greatly simplifies the necessary algebra compared to component calculations. Alternatively one could resort to the manifestly $\mathcal{N} = 2$ supersymmetric harmonic superspace approach, as in [10], in which the $\mathcal{N} = 4$ multiplet is realised putting together a $\mathcal{N} = 2$ vector multiplet and a hypermultiplet.

We prefer to work in the more familiar $\mathcal{N} = 1$ formalism. In this approach, the $\mathcal{N} = 4$ vector multiplet decomposes into three $\mathcal{N} = 1$ chiral multiplets Φ^I and one $\mathcal{N} = 1$ vector multiplet, all in the adjoint representation of the SU(N) gauge group. The lowest scalar components of the chiral multiplets are the ϕ^I introduced above.

Let us start by computing the superspace extension of $G^{(H)}(x_1, x_2, x_3, x_4)$ that we denote by $\Gamma^{(H)}(z_1, z_2, z_3, z_4)$, where $z_p = (x_p, \theta_p, \bar{\theta}_p)$ are the coordinates of $\mathcal{N} = 1$

[1]Notice that the presence of the factor g^2 in the normalisation of $\mathcal{K}_{\mathbf{84}}^{(11|22)}$ is dictated by the form of the supersymmetry transformations acting on the lowest component $\mathcal{K}_{\mathbf{1}}$ in eq. (3.5) and implies a shortening of the Konishi supermultiplet in the free field-theory limit [11].

[2]We thank Dan Freedman for pointing this out to us.

JHEP08(1999)020

superspace. As in the component field computations of (3.6), the only tree level contribution to

$$\Gamma^{(H)}(z_1, z_2, z_3, z_4) = \left\langle \mathrm{tr}\left[\left(\Phi^1\right)^2\right](z_1)\mathrm{tr}\left[\left(\Phi_1^\dagger\right)^2\right](z_2)\mathrm{tr}\left[\left(\Phi^2\right)^2\right](z_3)\mathrm{tr}\left[\left(\Phi_2^\dagger\right)^2\right](z_4)\right\rangle \tag{4.1}$$

comes from a disconnected super-diagram that grows as N^4 at large N. Because of colour contractions, the order g^2 correction to the disconnected diagram is trivially zero. In the Fermi-Feynman gauge, no corrections to the superfield propagators should be included [27] and there is a unique non-vanishing first order correction represented by a single connected diagram, in which the internal propagator corresponds to the exchange of a chiral superfield with "flavour" $I = 3$. Moreover, because of the non-renormalisation theorems for the two-point functions discussed in [20], there are no non-vanishing disconnected super-diagrams contributing at first or higher order.

Since we are dealing with Green functions of composite operators the result is necessarily depends on the θ-variables associated to each of the four external points. This is not in agreement with the results of calculations presented in [28]. However, if we restrict our attention to the lowest components of the chiral superfields, the expression in [28], though rather implicit, turns out to be correct. The explicit result is

$$G^{(H)}_{(1)}(x_1, x_2, x_3, x_4) = -\frac{g^2 N(N^2-1)}{2(2\pi)^{12} x_{12}^2 x_{34}^2} \int d^4x_0 \frac{1}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} \,. \tag{4.2}$$

Introducing Feynman parameters it can be expressed as

$$G^{(H)}_{(1)}(x_1, x_2, x_3, x_4) = -\frac{g^2 N(N^2-1)\pi^2}{2(2\pi)^{12} x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2} \, B(r,s) \,, \tag{4.3}$$

where $B(r, s)$ is a box-type integral given by

$$B(r,s) = \int \frac{d\beta_0 d\beta_1 d\beta_2 \; \delta(1-\beta_0-\beta_1-\beta_2)}{\beta_1\beta_2 + r\beta_0\beta_1 + s\beta_0\beta_2} \,. \tag{4.4}$$

As indicated, $B(r, s)$ depends only on the two independent conformally invariant cross ratios

$$r = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \,, \qquad s = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \,. \tag{4.5}$$

The result of the integration in (4.4) can be expressed as a combination of logarithms and dilogarithms as follows

$$\begin{aligned} B(r,s) = \frac{1}{\sqrt{p}} \Bigg\{ & \ln(r)\ln(s) - \left[\ln\left(\frac{r+s-1-\sqrt{p}}{2}\right)\right]^2 - \\ & -2\mathrm{Li}_2\left(\frac{2}{1+r-s+\sqrt{p}}\right) - 2\mathrm{Li}_2\left(\frac{2}{1-r+s+\sqrt{p}}\right)\Bigg\} \,, \end{aligned} \tag{4.6}$$

where $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ and[3]

$$p = 1 + r^2 + s^2 - 2r - 2s - 2rs\,. \tag{4.7}$$

Unlike correlation functions of elementary SYM fields, that are infra-red problematic and gauge-dependent (see e.g. [27, 29] for a recent analysis), the correlator (4.3) is well defined at non-coincident points and it has all the expected symmetry properties.

Notice that the representation of $B(r,s)$ given by (4.6) is valid only in the 'physical domain', i.e. in the region resulting from physically allowed choices of x_{pq}^2. In the euclidean regime it is defined by the condition $p \leq 0$. Although individual terms in (4.6) are complex, the complete function is real and positive in the 'physical domain'. Moreover the singularities corresponding to the three possible channels in the four-point function $G^{(H)}$ are located at $(r = 0, s = 1)$, $(r = 1, s = 0)$ and $(r = \infty, s = \infty)$.

Let us consider the limit $x_{12} \to 0$. The four-point function behaves as

$$x_{12} \to 0: \quad G^{(H)}_{(1)}(x_1, x_2, x_3, x_4) \to \frac{g^2 N (N^2-1)\pi^2}{(2\pi)^{12} x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2} \left(\frac{1}{2} \log \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} - 1 \right). \tag{4.8}$$

According to eq. (2.4), we can extract the one-loop contribution to the anomalous dimension of the Konishi multiplet as the coefficient of the logarithmic term, after dividing out the tree-level contribution

$$\langle \mathcal{K}_{\mathbf{1}}(x) \mathcal{K}_{\mathbf{1}}(y) \rangle_{(0)} = \frac{N^2-1}{3(2\pi)^4 (x-y)^4} \tag{4.9}$$

and taking into account the tree-level result

$$\langle \mathcal{C}^{11}(x_1) \mathcal{C}^{\dagger}_{11}(x_3) \mathcal{K}_{\mathbf{1}}(x_2) \rangle_{(0)} = \frac{N^2-1}{3(2\pi)^6 (x_{13})^2 (x_{12})^2 (x_{23})^2}\,. \tag{4.10}$$

We get in this way

$$\gamma_{\mathcal{K}}^{1-\mathrm{loop}}(g) = \frac{3}{16\pi^2} g^2 N\,, \tag{4.11}$$

in agreement with previous results [23]. Indeed the only other gauge-invariant operators of dimension two that can be exchanged are the protected operators $\mathcal{Q}_{\mathbf{20}}$, which cannot receive quantum corrections to their scale dimensions.

The subleading logarithmic singularity in this channel involves a rather complicated mixture of double-trace operators and superconformal descendants of the Konishi multiplet that will be discussed in detail in [30].

Let us briefly comment on the other channels. The limit $x_{13} \to 0$ in (4.3) exposes the u-channel (the t-channel is similar) and singles out the double-trace operator

[3]Notice the change of notation with respect to [3]: Δ there is p here.

JHEP08(1999)020

$\mathcal{D}^{(11|22)}$ defined in (3.15). As already observed, the $\mathcal{D}_{\mathbf{105}}$ component is protected from receiving corrections to OPE coefficients and anomalous dimension. This can be seen more clearly by considering the short distance behaviour of the one-loop contribution to the four-point function considered in [3]. The presence of logarithms in this channel imply that one or both operators of naive dimension $\Delta^{^{(0)}} = 4$ in the $\mathbf{84}$ in eqs. (3.17), (3.20) have anomalous dimension. Although $\mathcal{K}_{\mathbf{84}}$ alone could account for the logarithm in this channel, leaving $\widehat{\mathcal{D}}_{\mathbf{84}}$ with vanishing anomalous dimension at one-loop level, in order to settle this issue one has to consider another independent Green function [30]. For instance one can take the $\theta_1^2\overline{\theta}_2^2\theta_3^2\overline{\theta}_4^2$-component of the Green function (4.1) at order g^4

$$\langle \mathcal{E}^{11}(x_1)\mathcal{E}^{\dagger}_{11}(x_2)\mathcal{E}^{22}(x_3)\mathcal{E}^{\dagger}_{22}(x_4)\rangle \tag{4.12}$$

where, denoting by ψ^I and F^I the fermionic and auxiliary components of the Φ^I superfield, we have introduced the definition

$$\mathcal{E}^{IJ} = \mathrm{tr}(\psi^I\psi^J + \phi^I F^J + \phi^J F^I)\,. \tag{4.13}$$

$\mathcal{E}$ has protected dimension $\Delta = 3$ and belongs to the representation $\mathbf{6}_{+1}$ in the decomposition of the $\mathbf{10}$ of SU(4) with respect to SU(3) $\times$ U(1).

We would like to conclude this section with a general observation about the large N-behaviour of the anomalous dimensions of unprotected double-trace operators $\mathcal{D}$, which schematically we write as

$$\mathcal{D} = :\mathrm{tr}(\varphi\varphi)\ \mathrm{tr}(\varphi\varphi): \ . \tag{4.14}$$

Their one-loop anomalous dimensions are expected to be of the form

$$\gamma_{\mathcal{D}}^{1-\mathrm{loop}}(g) = \frac{c_N^{(1)}}{4\pi^2}\frac{g^2N}{N^2}\,, \tag{4.15}$$

where $c_N^{(1)}$ is a (possibly vanishing) constant of O(1) for large N. This follows by looking at the large N-behaviour of their tree-level two-point function

$$\langle \mathcal{D}^{\dagger}(x)\mathcal{D}(y)\rangle_{(0)} = \frac{a_N^{(0)}}{(2\pi)^8}\frac{N^4}{(x-y)^8}\,, \tag{4.16}$$

where again $a_N^{(0)}$ is a positive constant of O(1) in the large N limit, which depends on the specific choice of the operator. We expect the same pattern to persist at higher L-loops with

$$\gamma_{\mathcal{D}}^{L-\mathrm{loop}}(g) = \frac{c_N^{(L)}}{(4\pi^2)^L}\frac{(g^2N)^L}{N^2} \tag{4.17}$$

implying a vanishing of the perturbative value of $\gamma_{\mathcal{D}}$ in the large N limit at all orders.

Similar arguments apply to multi-trace operators. For instance, taking $\mathcal{M}_q = \mathrm{tr}(\varphi\varphi)^q$, one finds

$$\langle \mathcal{M}^\dagger(x)\mathcal{M}(y)\rangle_{_{(0)}} = \frac{b_N^{(0)}}{(2\pi)^{4q}}\frac{N^{2q}}{(x-y)^{4q}}\,, \tag{4.18}$$

where q denotes the number of colour traces in the definition of $\mathcal{M}$ and $b_N^{(0)}$ is a positive constant of O(1) for large N.

5. Non-perturbative instanton calculations

Relying on the remarkable properties of instanton calculus, the one-instanton contribution to the four-point function

$$G^{(V)}(x_1,x_2,x_3,x_4) = \langle \mathcal{C}^{11}(x_1)\mathcal{C}^\dagger_{11}(x_2)\mathcal{C}^{11}(x_3)\mathcal{C}^\dagger_{11}(x_4)\rangle \tag{5.1}$$

in $\mathcal{N}$=4 SYM theory with SU(2) gauge group has been computed (to lowest order in the coupling constant) in ref. [3]. The extension of that result to the case of SU(N) and to any instanton number in the large N limit has been given in [4].

Following the steps detailed in [3], we give here a closed-form expression for the one-instanton contribution to $G^{(H)}$ (eq. (3.6)).

We recall that in order to saturate the sixteen 'unlifted' [4] fermionic zero-modes present in the one-instanton background, the fields ϕ^I and $\phi^\dagger_I$ have to be 'replaced' with the expressions

$$\begin{aligned}\phi^I(x) \to \phi^I_{_{(0)}}(x) &= \frac{1}{2\sqrt{2}}\zeta^0\sigma^{\mu\nu}\zeta^I F_{\mu\nu}\,,\\ \phi^\dagger_I(x) \to \phi^\dagger_{I_{(0)}}(x) &= \frac{1}{4\sqrt{2}}\varepsilon_{IJK}\zeta^J\sigma^{\mu\nu}\zeta^K F_{\mu\nu}\,,\end{aligned} \tag{5.2}$$

where $F_{\mu\nu}$ is the one-instanton field-strength and $\zeta^A_\alpha = \eta^A_\alpha + \widehat{x}_{\alpha\dot\alpha}\overline{\epsilon}^{\dot\alpha A}$, $A = I, 0$, with $\widehat{x}_{\alpha\dot\alpha} = x_\mu(\sigma^\mu)_{\alpha\dot\alpha}$, are fermionic collective coordinates. The index $I = 1,2,3$ labels the fermionic zero modes in the $\mathcal{N} = 1$ chiral multiplets, the index 0 those associated to the $\mathcal{N} = 1$ gaugino. The instanton-induced 'scalar fields' (5.2) satisfy the scalar equation in the presence of the fermionic zero-modes. They correspond to the leading non-vanishing effective field configurations that result from the Wick contractions with the Yukawa terms obtained after expanding the exponential of the action until a number of fermion fields equal to the number of instanton induced fermionic zero modes are brought down. The scalar field configurations (5.2) can also be obtained directly from supersymmetry transformations by acting twice on $F_{\mu\nu}$ with the supersymmetry and superconformal transformations associated to ζ's, not preserved in the instanton background.

JHEP08(1999)020

After some Fierz transformations on the fermionic collective coordinates the fermionic integrations can be performed in a standard way [31] and yield

$$
\begin{aligned}
&G^{(H)}(x_1, x_2, x_3, x_4) = \\
&= \frac{3^4}{4\pi^{10}} g^8 e^{-\frac{8\pi^2}{g^2}} (e^{i\theta} + e^{-i\theta})\, x_{13}^2 x_{24}^2 x_{14}^2 x_{23}^2 \int \frac{d\rho_0 d^4 x_0}{\rho_0^5} \prod_{p=1}^{4} \left[\frac{\rho_0}{\rho_0^2 + (x_p - x_0)^2} \right]^4 , \qquad (5.3)
\end{aligned}
$$

where we have added up instanton and anti-instanton contributions.[4] Up to an overall conformal invariant factor $(x_{14}^2 x_{23}^2)/(x_{13}^2 x_{24}^2) = s$, the result (5.3) coincides with the one found for (5.1) in [3]. Exactly the same relation, $G^{(H)}(x_1, x_2, x_3, x_4) = sG^{(V)}(x_1, x_2, x_3, x_4)$ is valid at one-loop [10, 30].

Apart from numerical factors, the same result is obtained for any SU(N) gauge group [4]. This follows from the fact that the extra 'non-geometric' $8N-16$ fermionic zero-modes, present for $N > 2$, turn out to be effectively lifted, as they appear in quadrilinear terms in the full SYM action. The lifting can be most easily understood as the result of the functional integration over the scalar field fluctuations around the classical configuration ($\phi = 0$), noticing that, in the semiclassical approximation in which we are working, fermionic fields are to be effectively replaced by their zero mode expression. Bosonising the lifted fermionic zero-modes *á la* Hubbard-Stratonovich and performing the extra bosonic integrations associated to the new bosonic zero modes produces the extra factor $\Gamma(N - \frac{1}{2})/\Gamma(N-1)$ in front of (5.3) that precisely matches the AdS-inspired expectation, $\sqrt{N}$, in the large N limit [4].

Higher instanton numbers may be exactly taken into account in the large N limit, because a saddle-point approximation is possible. Contrary to naive expectations, the instanton gas is far from being dilute in the sense that at large N the K instantons tend to "attract" each other and share the same position, x_0, and size, ρ_0. Still they lie in K commuting SU(2) subgroups of the SU(N) gauge group. Properly taking into account the rôle of the lifted and bosonised fermionic zero-modes, one can show that the situation is as if the K instanton moduli space were a single copy of $AdS_5 \times S^5$ [4]. The same situation prevails in type IIB D-instanton computations and all K-dependent factors on the SYM side felicitously reproduce the type IIB D-instanton induced terms [33, 34, 3].

In summary, up to computable overall constants, the one-instanton contribution to the four-point functions under consideration is the same for any SU(N) gauge group and, in the large N limit, every instanton number contributes the same

[4]It should be noted that for this class of four-point functions instanton and anti-instanton contributions are simply complex conjugate to one another. On the contrary, at lowest order in g, correlation functions of protected operators that violate the external automorphism $\mathrm{U}(1)_B$ of the $\mathcal{N} = 4$ superconformal algebra, whose AdS counterpart is the anomalous $\mathrm{U}(1)_B$ symmetry of the type IIB supergravity, only receive contribution from instantonic configurations with topological charge of a given sign [3, 32].

JHEP08(1999)020

expression. In the large N limit, the sum over K gives rise to an overall non-holomorphic function of the complexified gauge coupling τ, which coincides with the non-perturbative D-instanton contribution to the coefficient, $f_4(\tau,\bar\tau)$, appearing in front of the $\mathcal{R}^4$ terms in the type IIB low-energy effective action [33].

Going back to eq. (5.3), we notice that the x_0 integration resembles that of a standard Feynman diagram with momenta replaced by position differences and can be performed either by expressing it in terms of derivatives of the box-integral as

$$\int \frac{d\rho_0 d^4x_0}{\rho_0^5}\prod_{p=1}^{4}\left[\frac{\rho_0}{\rho_0^2+(x_p-x_0)^2}\right]^4 = \frac{5\pi^2}{108}\prod_{p<q}\frac{\partial}{\partial x_{pq}^2}\int\prod_p d\alpha_p \frac{\delta\left(1-\sum_q\alpha_q\right)}{(\sum_{p,q}\alpha_p\alpha_q x_{pq}^2)^2}\,, \tag{5.4}$$

or by introducing the alternative parameterisation

$$\int \frac{d\rho_0 d^4x_0}{\rho_0^5}\prod_{p=1}^{4}\left[\frac{\rho_0}{\rho_0^2+(x_p-x_0)^2}\right]^4 = \frac{5\pi^2}{3}\frac{1}{x_{13}^8x_{24}^8}I(r,s)\,, \tag{5.5}$$

where I is the integral

$$I(r,s) = \int d\beta_0 d\beta_1 d\beta_2 \frac{\beta_0^3\beta_1^3\beta_2^3\ \delta(1-\beta_0-\beta_1-\beta_2)}{(\beta_1\beta_2+r\beta_0\beta_1+s\beta_0\beta_2)^4}\,. \tag{5.6}$$

After a rather lengthy calculation both approaches yield

$$I(r,s) = \frac{P_0(r,s)}{p^6}\ B(r,s) + \frac{P_1(r,s)}{3p^6}\ \ln(r) + \frac{P_1(s,r)}{3p^6}\ \ln(s) + \frac{P_2(r,s)}{18p^5}\,, \tag{5.7}$$

where $B(r,s)$ is given by (4.6), p is defined in (4.7) and $P_i(r,s)$ are polynomials in the cross ratios r and s introduced in (4.5). Their explicit expressions are

$$\begin{aligned}P_0(r,s) = {} & 1+3s+3r-36r^5s^4-498rs^5+3948r^3s^3-498r^5s-1338r^3s^4+62r^3s^6-\\ & -36r^4s^5+1050r^4s^4+1512r^2s^2+3r^8s-30r^2s^7-30r^7s^2-1338r^4s^3+\\ & +1512r^5s^2+1512r^2s^5+1050r^4s-498r^5s^3-498r^3s^5+62r^6s^3+3rs^8-\\ & -144r^6s-144s^6r-144s^2r+1050rs^4-498r^3s+114rs^7-1338r^2s^3-\\ & -144r^6s^2+114r^7s-144r^2s^6-498rs^3-1338r^4s^2-144r^2s-1338r^3s^2+\\ & +114rs-1338r^2s^4-30s^2+62r^3-36r^5-36r^4+3s^8-30s^7+62s^3-\\ & -30r^7+3r^8+s^9+r^9+62r^6+62s^6-36s^5-36s^4-30r^2\end{aligned}$$

$$\begin{aligned}P_1(r,s) = {} & 11-28s+62r-2142rs^5+5132r^3s^3+1170r^5s-4894r^3s^4+215r^4s^4+\\ & +4491r^2s^2+3404r^4s^3+1170r^5s^2+972r^2s^5+3404r^4s-646r^5s^3+\\ & +362r^3s^5-1180r^6s+590s^6r-2142s^2r+1490rs^4-4894r^3s+62rs^7-\\ & -10264r^2s^3+383r^6s^2-34r^7s-331r^2s^6+1490rs^3-8982r^4s^2+972r^2s+\\ & +5132r^3s^2+590rs+4491r^2s^4-52s^2+362r^3-646r^5+215r^4+11s^8-\\ & -28s^7+284s^3-34r^7-22r^8+383r^6-52s^6+284s^5-430s^4-331r^2\end{aligned}$$

$$
\begin{aligned}
P_2(r,s) = {} & -114s - 114r - 114rs^5 + 2404r^3s^3 - 114r^5s + 15402r^2s^2 + 4734r^4s - \\
& - 4620s^2r + 4734rs^4 - 4620r^3s - 4620r^2s^3 - 4620rs^3 - 1281r^4s^2 - \\
& - 4620r^2s - 4620r^3s^2 + 4734rs - 1281r^2s^4 - 1281s^2 + 2404r^3 - 114r^5 - \\
& - 1281r^4 + 2404s^3 + 193r^6 + 193s^6 - 114s^5 - 1281s^4 - 1281r^2 + 193\,.
\end{aligned}
$$

Having obtained a closed expression for $G^{(H)}$, one can unambiguously study the behaviour when any two points are taken close to one another, $x_{pq} \to 0$. An explicit calculation shows that $I(r,s)$ has singularities only at the points $(r=0, s=1)$, $(r=1, s=0)$ and $(r=\infty, s=\infty)$. Taking into account the factor, $x_{13}^2x_{24}^2x_{14}^2x_{23}^2$, present in the complete expression of $G^{(H)}$, one finds that the only case in which there is actually a singularity is when any $\mathcal{C}$ approaches its conjugate $\mathcal{C}^\dagger$, which corresponds either to the limit $x_{12} \to 0$ or to $x_{34} \to 0$. In terms of the conformal cross-ratios s and r introduced before, both limits, in fact, correspond to $r \to 0$, $s \to 1$, where $I(r,s)$ develops a logarithmic singularity. Notice that individual terms in $I(r,s)$ have pole-type singularities (of degree up to seven), but in the sum only a logarithmic singularity survives.

In the limit $x_{12} \to 0$, the one-instanton plus one anti-instanton contribution to the four-point function behaves as

$$
\begin{aligned}
x_{12} \to 0: \quad & G^{(H)}(x_1,x_2,x_3,x_4) \to \\
& \to \frac{15}{64\pi^8}\, g^8\, e^{-\frac{8\pi^2}{g^2}} (e^{i\theta} + e^{-i\theta})\, \frac{1}{x_{13}^4 x_{14}^4} \left(-\frac{2706}{1225} - \frac{36}{35} \log \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \right).
\end{aligned} \tag{5.8}
$$

The absence of a term of the type $\log(x_{12}^2)/x_{12}^2$ implies that there is no one-instanton contribution to the anomalous dimension of the scalar $\mathcal{K}_1$. By supersymmetry arguments this result extends to the whole Konishi multiplet. The absence of a term of the type $1/x_{12}^2$ confirms the vanishing of any quantum correction to the two-point function of the protected gauge-invariant operators of dimension 2, $\mathcal{Q}_{\mathbf{20}}$. The leading logarithmic singularity present in (5.8) comes from a mixture of double-trace operators of naive dimension $\Delta^{(0)} = 4$. Extracting the non-perturbative contribution to the anomalous dimension and OPE coefficients of all these operators is beyond the scope of this paper and will be discussed elsewhere [30]. Let us stress, however, that the non-vanishing coefficient of the logarithm in (5.8) implies that at least one double trace scalar operator of tree-level dimension $\Delta^{(0)} = 4$ has a non-vanishing one instanton correction to its anomalous dimension.

In the other two channels the vanishing of the factor $x_{13}^2x_{24}^2x_{14}^2x_{23}^2$ leads to a vanishing result. This immediately implies that only operators of dimension higher than four can contribute in these channels. Notice that performing a similar OPE analysis of the four-point function computed in [3] leads one to conclude that $\mathcal{D}_{\mathbf{105}}$ cannot be exchanged in the u-channel. The remaining operators of dimension four that can be exchanged in this channel belong to the representation $\mathbf{84}$ of SU(4) and are the component of the Konishi multiplet $\mathcal{K}_{\mathbf{84}}$ in eq. (3.20) and the double-trace operator $\widehat{\mathcal{D}}_{\mathbf{84}}$

introduced at the end of section 3. Since, as we have demonstrated, the one-instanton correction to the anomalous dimension of $\mathcal{K}_{\mathbf{84}}$ is zero, one can immediately conclude that the one-instanton correction to the anomalous dimension of $\widehat{\mathcal{D}}_{\mathbf{84}}$ is zero as well.

Let us also note that in general the one (anti-)instanton contributions to the anomalous dimensions of double-trace operators are expected to be of the form

$$\gamma_{\mathcal{D}}^{1-\mathrm{inst}} = \frac{C_N^{K=\pm 1}}{N^4} \frac{\Gamma(N-\frac{1}{2})}{\Gamma(N-1)} e^{-\frac{8\pi^2}{g^2}} (e^{i\theta} + e^{-i\theta}) \,, \tag{5.9}$$

where $C_N^{K=\pm 1}$ is a constant of O(1) for large N. Once again the scaling of the denominator with N^4 is determined by the tree-level normalisation of the double-trace operators.

The observation that neither operators in the supercurrent multiplet nor operators in the Konishi multiplet are exchanged in any of the intermediate channels at the one (anti-) instanton level, implies that the known non-renormalisation theorems for three-point functions of protected operators [20] can be generalised at non-perturbative level. This result should not be considered in contradiction with the expected S-duality of the $\mathcal{N} = 4$ SYM theory, as S-duality is not a pointwise symmetry relating individual operators. For instance, string states, such as those corresponding to the Konishi multiplet, get transformed into D-string states or to other dyonic states, so that a significant reshuffling among operators should take place under S-duality. Slightly at variant with respect to [32], we only expect that the whole spectrum of (anomalous) dimensions and the whole set of OPE coefficients should be invariant under S-duality.

6. Conclusions and comments on the AdS/SCFT correspondence

The outcome of our analysis is that the short distance logarithmic behaviour of the Green functions in $\mathcal{N} = 4$ SYM theory in the (super)conformal phase is a manifestation of the presence of non vanishing anomalous dimensions of unprotected operators. A highly non-trivial, but technically very involved, check of the validity of our interpretation, would consist in showing that the next order corrections give rise to square logarithms with coefficients that are precisely given by the squares of the first order coefficients as demanded for the exponentiation of the logarithmic terms.

The presence of logarithms might be thought to spoil the perturbative finiteness properties of $\mathcal{N} = 4$ SYM theory that have been known for a long time. Although, as we have argued in this paper, this is not true, the analysis presented in [27] has pointed out a number of problems in the perturbative computation of Green functions of elementary (super) fields. The calculations performed in [27] have clarified many subtleties related to the gauge-fixing procedure of the theory, both in the description in terms of component fields and using the $\mathcal{N} = 1$ superfield formalism.

The appearance of UV divergences in off-shell propagators, when the Wess–Zumino (WZ) gauge is exploited, has been discussed and it has been shown that relaxing the WZ gauge choice makes UV divergences disappear.

The analysis of [27, 29] has also shown that in Green functions of elementary (super)fields there are IR divergences even for non-exceptional values of the momenta. These are in fact gauge artifacts and are due to the $1/k^4$ behaviour of the propagator of the lowest component of the vector superfield. This problematic term in the superfield propagator disappears in the (supersymmetric generalisation of the) Fermi–Feynman gauge and in the case the WZ gauge is employed. It would be interesting to obtain an explicit check of the cancellation of IR divergences in gauge invariant Green functions for an arbitrary gauge-choice, even without restricting the theory to the WZ gauge.

Finally let us discuss the bearing of our results on the validity and the meaning of the Maldacena conjecture. We have shown by scaling arguments that all the perturbative and non-perturbative contributions to the anomalous dimensions of multi-trace operators actually vanish in the large N limit. The situation for the Konishi multiplet is different, because its anomalous dimension does not seem to vanish in the large N limit. Actually on the basis of the AdS/SCFT correspondence it is suggested that it grows as large as $N^{1/4}$, eventually decoupling from the operator algebra.

Short-distance logarithmic singularities have been shown to appear in genuine AdS supergravity calculations [8, 35]. These logarithms are expected to be related to the exchange of multi-particle bound states in the bulk which are in correspondence with unprotected multi-trace operators on the boundary. In view of the vanishing of the anomalous dimensions of multi-trace operators that we find at large N, it is not obvious to us what is the relation between the logarithms seen in perturbative and non-perturbative $\mathcal{N} = 4$ SYM calculations and the logarithms identified in [8, 35] despite their suggestive similarity [36].

Acknowledgments

We would like to thank L. Andrianopoli, D. Anselmi, M. Bochicchio, S. Ferrara, P. Fré, M.B. Green, R. Musto, F. Nicodemi, R. Pettorino, A.M. Polyakov, D. Polyakov, A. Sagnotti, E. Sokatchev, M. Testa, I.T. Todorov and K. Yoshida for useful discussions. Ya.S.S. would like to thank the Physics Department and I.N.F.N. Section at Università di Roma "Tor Vergata" for hospitality and financial support. The results of this investigation have been preliminarily presented by M.B. at the Meeting "Advances in Theoretical Physics" held in Vietri, Italy, April 1999 and by S.K. at the "Triangular Meeting on Quantum Field Theory in Particle Physics" held in Utrecht, Holland, May 1999.

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